Chapter 1

A Review of Analytical Mechanics

1.1 Introduction

These lecture notes cover the third course in Classical Mechanics, taught at MIT since the Fall of 2012 by Professor Stewart to advanced undergraduates (course 8.09) as well as to graduate students (course 8.309). In the prerequisite classical mechanics II course the students are taught both Lagrangian and Hamiltonian dynamics, including Kepler bound motion and central force scattering, and the basic ideas of canonical transformations. This course briefly reviews the needed concepts, but assumes some familiarity with these ideas. References used for this course include

- Goldstein, Poole & Safko, *Classical Mechanics*, 3rd edition.
- Landau and Lifshitz vol.6, *Fluid Mechanics*. Symon, *Mechanics* for reading material on non-viscous fluids.
- Strogatz, *Nonlinear Dynamics and Chaos.*
- Review: Landau & Lifshitz vol.1, *Mechanics*. (Typically used for the prerequisite Classical Mechanics II course and hence useful here for review)

1.2 Lagrangian & Hamiltonian Mechanics

Newtonian Mechanics

In Newtonian mechanics, the dynamics of a system of N particles are determined by solving for their coordinate trajectories as a function of time. This can be done through the usual vector spatial coordinates $\mathbf{r}_i(t)$ for $i \in \{1, \ldots, N\}$, or with generalized coordinates $q_i(t)$ for $i \in \{1, \ldots, 3N\}$ in 3-dimensional space; generalized coordinates could be angles, et cetera.

Velocities are represented through $\mathbf{v}_i \equiv \dot{\mathbf{r}}_i$ for spatial coordinates, or through $\dot{q}_i$ for generalized coordinates. Note that dots above a symbol will always denote the total time derivative $\frac{d}{dt}$. Momenta are likewise either Newtonian $\mathbf{p}_i = m_i \mathbf{v}_i$ or generalized p_i.

For a fixed set of masses m_i Newton's 2^{nd}law can be expressed in 2 equivalent ways:

1. It can be expressed as N second-order equations $\mathbf{F}_i = \frac{d}{dt}(m_i \dot{\mathbf{r}}_i)$ with $2N$ boundary conditions given in $\mathbf{r}_i(0)$ and $\dot{\mathbf{r}}_i(0)$. The problem then becomes one of determining the N vector variables $\mathbf{r}_i(t)$.

2. It can also be expressed as an equivalent set of $2N$ 1^{st} order equations $\mathbf{F}_i = \dot{\mathbf{p}}_i$ & $\mathbf{p}_i/m_i = \dot{\mathbf{r}}_i$ with $2N$ boundary conditions given in $\mathbf{r}_i(0)$ and $\mathbf{p}_i(0)$. The problem then becomes one of determining the $2N$ vector variables $\mathbf{r}_i(t)$ and $\mathbf{p}_i(t)$.

Note that $\mathbf{F} = m\mathbf{a}$ holds in *inertial frames*. These are frames where the motion of a particle not subject to forces is in a straight line with constant velocity. The converse does not hold. Inertial frames describe time and space homogeneously (invariant to displacements), isotropically (invariant to rotations), and in a time independent manner. Noninertial frames also generically have fictitious "forces", such as the centrifugal and Coriolis effects. (Inertial frames also play a key role in special relativity. In general relativity the concept of inertial frames is replaced by that of geodesic motion.)

The existence of an inertial frame is a useful approximation for working out the dynamics of particles, and non-inertial terms can often be included as perturbative corrections. Examples of approximate inertial frames are that of a fixed Earth, or better yet, of fixed stars. We can still test for how noninertial we are by looking for fictitious forces that (a) may point back to an origin with no source for the force or (b) behave in a non-standard fashion in different frames (i.e. they transform in a strange manner when going between different frames).

We will use primes will denote coordinate transformations. If $\mathbf{r}$ is measured in an inertial frame S, and $\mathbf{r}'$ is measured in frame S' with relation to S by a transformation $\mathbf{r}' = \mathbf{f}(\mathbf{r}, t)$, then S' is inertial iff $\ddot{\mathbf{r}} = 0 \leftrightarrow \ddot{\mathbf{r}}' = 0$. This is solved by the Galilean transformations,

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \mathbf{v}_0 t \\ t' &= t, \end{aligned}$$

which preserves the inertiality of frames, with $\mathbf{F} = m\ddot{\mathbf{r}}$ and $\mathbf{F}' = m\ddot{\mathbf{r}}'$ implying each other. Galilean transformations are the non-relativistic limit, $v \ll c$, of Lorentz transformations which preserve inertial frames in special relativity. A few examples related to the concepts of inertial frames are:

1. In a rotating frame, the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If $\theta = \omega t$ for some constant ω, then $\ddot{\mathbf{r}} = 0$ still gives $\ddot{\mathbf{r}}' \neq 0$, so the primed frame is noninertial.

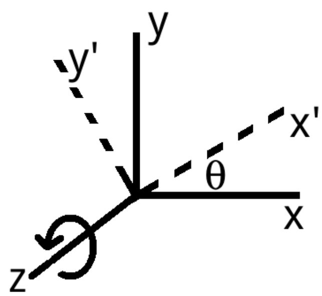

Figure 1.1: Frame rotated by an angle θ

2. In polar coordinates, $\mathbf{r} = r\hat{r}$, gives

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}, \qquad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} \tag{1.1}$$

and thus

$$\ddot{\mathbf{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\left(\ddot{\theta}\hat{\theta} - \dot{\theta}^2\hat{r}\right) . \tag{1.2}$$

Even if $\ddot{\mathbf{r}} = 0$ we can still have $\ddot{r} \neq 0$ and $\ddot{\theta} \neq 0$, and we can not in general form a simple Newtonian force law equation $m\ddot{q} = F_q$ for each of these coordinates. This is different than the first example, since here we are picking coordinates rather than changing the reference frame, so to remind ourselves about their behavior we will call these "non-inertial coordinates" (which we may for example decide to use in an inertial frame). In general, curvilinear coordinates are non-inertial.

Lagrangian Mechanics

In Lagrangian mechanics, the key function is the Lagrangian

$$L = L(q, \dot{q}, t). \tag{1.3}$$

Here, $q = (q_1, \ldots, q_N)$ and likewise $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_N)$. We are now letting N denote the number of scalar (rather than vector) variables, and will often use the short form to denote dependence on these variables, as in Eq. (1.3). Typically we can write $L = T - V$ where T is the kinetic energy and V is the potential energy. In the simplest cases, $T = T(\dot{q})$ and $V = V(q)$, but we also allow the more general possibility that $T = T(q, \dot{q}, t)$ and $V = V(q, \dot{q}, t)$. It turns out, as we will discuss later, that even this generalization does not describe all possible classical mechanics problems.

The solution to a given mechanical problem is obtained by solving a set of N second-order differential equations known as *Euler-Lagrange equations of motion*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1.4}$$

These equations involve $\ddot{q}_i$, and reproduce the Newtonian equations $\mathbf{F} = m\mathbf{a}$. The principle of stationary action (Hamilton's principle),

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0, \tag{1.5}$$

is the starting point for deriving the Euler-Lagrange equations. Although you have covered the Calculus of Variations in an earlier course on Classical Mechanics, we will review the main ideas in Section 1.5.

There are several advantages to working with the Lagrangian formulation, including

1. It is easier to work with the scalars T and V rather than vectors like $\mathbf{F}$.

2. The same formula in equation (1.4) holds true regardless of the choice of coordinates. To demonstrate this, let us consider new coordinates

$$Q_i = Q_i(q_1, \ldots, q_N, t). \tag{1.6}$$

This particular sort of transformation is called a *point transformation*. Defining the new Lagrangian by

$$L' = L'(Q, \dot{Q}, t) = L(q, \dot{q}, t), \tag{1.7}$$

we claim that the equations of motion are simply

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}_i}\right) - \frac{\partial L'}{\partial Q_i} = 0. \tag{1.8}$$

Proof: (for $N = 1$, since the generalization is straightforward)
Given $L'(Q, \dot{Q}, t) = L(q, \dot{q}, t)$ with $Q = Q(q, t)$ then

$$\dot{Q} = \frac{d}{dt}Q(q, t) = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial t}. \tag{1.9}$$

Therefore

$$\frac{\partial \dot{Q}}{\partial \dot{q}} = \frac{\partial Q}{\partial q}, \tag{1.10}$$

a result that we will use again in the future. Then

$$\begin{aligned} \frac{\partial L}{\partial q} &= \frac{\partial L'}{\partial q} = \frac{\partial L'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q}, \\ \frac{\partial L}{\partial \dot{q}} &= \frac{\partial L'}{\partial \dot{q}} = \frac{\partial L'}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial \dot{q}} = \frac{\partial L'}{\partial \dot{Q}}\frac{\partial Q}{\partial q}\,. \end{aligned} \tag{1.11}$$

Since $\frac{\partial Q}{\partial \dot{q}} = 0$ there is no term $\frac{\partial L'}{\partial Q}\frac{\partial Q}{\partial \dot{q}}$ in the last line.
Plugging these results into $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$ gives

$$\begin{aligned} 0 &= \left[\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\right)\frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}}\frac{d}{dt}\left(\frac{\partial Q}{\partial q}\right)\right] - \left[\frac{\partial L'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q}\right] \\ &= \left[\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\right) - \frac{\partial L'}{\partial Q}\right]\frac{\partial Q}{\partial q}, \end{aligned} \tag{1.12}$$

since $\frac{d}{dt}\frac{\partial Q}{\partial q} = (\dot{q}\frac{\partial}{\partial q} + \frac{\partial}{\partial t})\frac{\partial Q}{\partial q} = \frac{\partial}{\partial q}(\dot{q}\frac{\partial}{\partial q} + \frac{\partial}{\partial t})Q = \frac{\partial \dot{Q}}{\partial q}$ so that the second and fourth terms cancel. Finally for non-trivial transformation where $\frac{\partial Q}{\partial q} \neq 0$ we have, as expected,

$$0 = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\right) - \frac{\partial L'}{\partial Q}\,. \tag{1.13}$$

Note two things:

- This implies we can freely use the Euler-Lagrange equations for noninertial coordinates.
- We can formulate L in whatever coordinates are easiest, and then change to convenient variables that better describe the symmetry of a system (for example, Cartesian to spherical).

3. Continuing our list of advantages for using L, we note that it is also easy to incorporate *constraints*. Examples include a mass constrained to a surface or a disk rolling without slipping. Often when using L we can avoid discussing forces of constraint (for example, the force normal to the surface).

Lets discuss the last point in more detail (we will also continue to discuss it in the next section). The method for many problems with constraints is to simply make a good choice for the generalized coordinates to use for the Lagrangian, picking $N - k$ independent variables q_i for a system with k constraints.
Example: For a bead on a helix as in Fig. 1.2 we only need one variable, $q_1 = z$.
Example: A mass m_2 attached by a massless pendulum to a horizontally sliding mass m_1 as in Fig. 1.3, can be described with two variables $q_1 = x$ and $q_2 = \theta$.
Example: As an example using non-inertial coordinates consider a potential $V = V(r, \theta)$ in polar coordinates for a fixed mass m at position $\mathbf{r} = r\hat{r}$. Since $\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ we have $T = \frac{m}{2}\dot{\mathbf{r}}^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$, giving

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r, \theta). \tag{1.14}$$